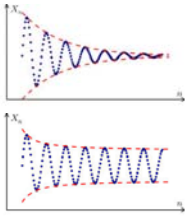
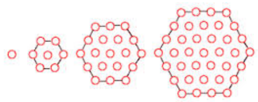


Growth Models



Sequences and Population Sequences

2, 4, 8, 16, 32
·2 ·2 ·2 ·2

Sequences

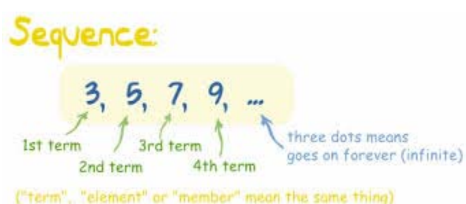
In mathematics the word *sequence* has a very specific meaning:

A **sequence** is an infinite, ordered list of numbers.
In principle, the numbers can be any type of number: positive, negative, zero, rational, or irrational.

Sequences

The individual numbers in a sequence are called the **terms** of the sequence.

(also referred to as **elements** or **members**)



Sequences

The simplest way to describe a sequence is using a list format—start writing the terms of the sequence, in order, separated by commas.

The list, however, is infinite, so at some point one has to stop writing. At that point, a ". . ." is added as a symbolic way of saying "and so on."

For lack of a better term, we will call this the **infinite list** description of the sequence.

Sequences

How many terms should we write at the front end before we appeal to the ". . ."?

This is a subjective decision, but the idea is to write enough terms so that a reasonable third party looking at the sequence can figure out how the sequence continues.

No matter what we do, the ". . ." is always a leap of faith, and we should strive to make that leap as small as possible. Some sequences become clear with four or five terms, others take more.

Example: How Many Terms are Enough?

Consider the sequence that starts with 1, 2, 4, 8, 16, 32, ...

We could have continued writing down terms, but it seems reasonable to assume that at this point most people would agree that the sequence continues with 64, 128, ...

The leap of faith here is small.

Example: How Many Terms are Enough?

Consider the sequence that starts with 3, 5, 7, ... Are there enough terms here so that we can figure out what comes next?

A good guess is that the sequence continues with 9, 11, 13, . . . but this is not the only reasonable guess.

Perhaps the sequence is intending to describe the odd prime numbers, and in that case the next three terms of the sequence would be 11, 13, 17, ...

Population Sequences

For the rest of this chapter we will focus on special types of sequences called *population sequences*.

For starters, let's clarify the meaning of the word **population**.

In its original meaning, the word refers to human populations (the Latin root of the word is *populus*, which means "people") but over time the scope of the word has been expanded to apply to many other "things"—animals, bacteria, viruses, Web sites, plastic bags, money, etc.

Population Sequences

The main characteristic shared by all these "things" is that their quantities change over time, and to track the ebb and flow of these changes we use a *population sequence*.

A **population sequence** describes the size of a population as it changes over time, measured in discrete time intervals.

A population sequence starts with an *initial* population (you have to start somewhere), and it is customary to think of the start as time zero.

Population Sequences

The size of the population at time zero is the first term of the population sequence. After some time goes by (it may be years, hours, seconds, or even nanoseconds), there is a "change" in the population—up, down, or it may even stay unchanged.

We call this change a *transition*, and the population after the first transition is the *first generation*.

Sequence Notation

The main lesson to be drawn from the example is that describing a sequence using an infinite list is simple and convenient, but it doesn't work all that well with the more exotic sequences.

Are there other ways? Yes.

Before we get to them, we introduce some useful notation for sequences

Sequence Notation

A generic sequence can be written in infinite list form as

$A_1, A_2, A_3, A_4, A_5, \ldots$

The A is a variable representing a symbolic name for the sequence.

Each term of the sequence is described by the sequence name and a numerical subscript that represents the position of the term in the sequence.

Sequence Notation

You may think of the subscript as the "address" of the term.

This notation makes it possible to conveniently describe any term by its position in the sequence: A_{10} represents the 10^{th} term, A_{100} represents the 100th term, and A_N represents a term in a generic position N in the sequence.

Sequence Notation

The aforementioned notation makes it possible to describe some sequences by just giving an **explicit formula** for the generic Nth term of the sequence.

That formula then is used with $N = 1$ for the first term, $N = 2$ for the second term, and so on.

Population Growth Model

The Linear Growth Model

Linear Growth

A population grows according to a **linear growth** model if in each generation the population changes by a constant amount.

When a population grows according to a linear growth model, that population grows *linearly*, and the population sequence is called an *arithmetic sequence*.

Linear Growth

Linear growth and arithmetic sequences go hand in hand, but they are not synonymous.

Linear growth is a term we use to describe a special type of population growth, while an *arithmetic sequence* is an abstract concept that describes a special type of number sequence.

Example: How Much Garbage Can We Take?

The city of Cleansburg is considering a new law that would restrict the monthly amount of garbage allowed to be dumped in the local landfill to a maximum of 120 tons a month.

There is a concern among local officials that unless this restriction on dumping is imposed, the landfill will reach its maximum capacity of 20,000 tons in a few years.

Example: How Much Garbage Can We Take?

Currently, there are 8000 tons of garbage in the landfill.

Suppose the law is passed right now, and the landfill collects the maximum allowed (120 tons) of garbage each month from here on.

(a) How much garbage will there be in the landfill five years from now?

(a) How long would it take the landfill to reach its maximum capacity of 20,000 tons?

Example: How Much Garbage Can We Take?

We can answer these questions by modeling the amount of garbage in the landfill as a population that grows according to a linear growth model.

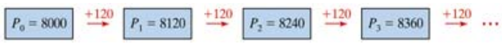
A very simple way to think of the growth of the garbage population is the following:

> Start with an initial population of P_0 = 8000 tons and *each month add 120 tons to whatever the garbage population was in the previous month.*

Example: How Much Garbage Can We Take?

This formulation gives the *recursive* formula $P_N = P_{N-1} + 120$, with $P_0 = 8000$ to get things started.

The figure illustrates the first few terms of the population sequence based on the recursive formula.



Example: How Much Garbage Can We Take?

For the purposes of answering the questions posed at the start of this example, the recursive formula is not particularly convenient.

Five years, for example, equals 60 months, and we would prefer to find the value of P_{60} without having to compute the first 59 terms in the sequence.

Example: How Much Garbage Can We Take?

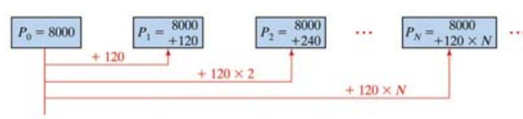
We can get a nice explicit formula for the growth of the garbage population using a slightly different interpretation:

> *In any given month N, the amount of garbage in the landfill equals the original 8000 tons plus 120 tons for each month that has passed.*

This formulation gives the *explicit* formula $P_N = 8000 + 120 * N$.

Example: How Much Garbage Can We Take?

The figure illustrates the growth of the population viewed in terms of the explicit formula.



Example: How Much Garbage Can We Take?

The explicit formula $P_N = 8000 + 120 * N$ will allow us to quickly answer the two questions raised at the start of the example.

(a) After five years (60 months), the garbage population in the landfill is given by $P_{60} = 8000 + 120 * 60 = 8000 + 7200 = 15{,}200$.

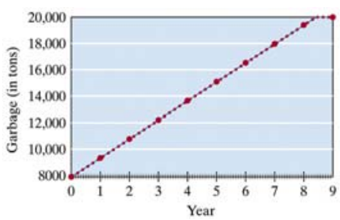
(b) If X represents the month the landfill reaches its maximum capacity of 20,000, then $20{,}000 = 8000 + 120X$.

Example: How Much Garbage Can We Take?

Solving for X gives $X = 100$ months. The landfill will be maxed out 8 years and 4 months from now.

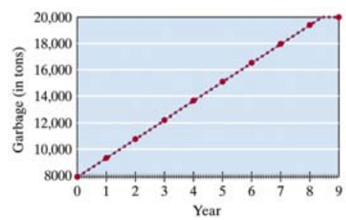
Example: How Much Garbage Can We Take?

The line graph in shows the projections for the garbage population in the landfill until the landfill reaches its maximum capacity.



Example: How Much Garbage Can We Take?

Not surprisingly, the line graph forms a straight line. This is always true in a linear growth model (and the reason for the name linear)—the growth of the population follows a straight line.



The Arithmetic Sum Formula

Suppose you are given an arithmetic sequence—say, 5, 8, 11, 14, 17, . . .—and you are asked to add a few of its terms.

How would you do it?

When it's just a very few, you would probably just add them term by term: 5 + 8 + 11 + 14 +…, but what if you were asked to add lots of terms—say the first 500 terms—of the sequence?

The Arithmetic Sum Formula

Adding 500 numbers, even with a calculator, does not seem like a very enticing idea.

Fortunately, there is a nice trick that allows us to easily add any number of consecutive terms in any arithmetic sequence.

Before giving the general formula, let's see the trick in action.

Example: Adding the first 500 terms of 5, 8, 11, 14, . . .

Since a population sequence starts with P_0, the first 500 terms are $P_0, P_1, P_2, \ldots, P_{499}$.

In the particular case of the sequence 5, 8, 11, 14, 17, . . . we have $P_0 = 5$, $d = 3$, and $P_{499} = 5 + 3 * 499 = 1502$.

The sum we want to find is

$S = 5 + 8 + 11 + \ldots + 1496 + 1499 + 1502$.

Example: Adding the first 500 terms of 5, 8, 11, 14, . . .

Now here comes the trick: (1) write the sum in the normal way, (2) below the first sum, write the sum again but do it backwards (and make sure the+signs are lined up), (3) add the columns, term by term.

In our case, we get

(1) $S = 5 + 8 + 11 + \ldots + 1496 + 1499 + 1502$.

(2) $S = 1502 + 1499 + 1496 + \ldots + 11 + 8 + 5$.

(3) $2S = 1507 + 1507 + 1507 + \ldots + 1507 + 1507 + 1507$.

Example: Adding the first 500 terms of 5, 8, 11, 14, . . .

The key is that what happened in (3) is no coincidence. In each column we get the same number:

1507 = 5 + 1502 = (starting term) +(ending term). Rewriting (3) as $2S = 500 * 1507$ and solving for S gives the sum we want: $S = (1507 * 500)/2 = 376{,}750$.

Example: Adding the first 500 terms of 5, 8, 11, 14, . . .

We will now generalize the trick we used in the preceding example. In the solution $S = (1507 * 500)/2$ the 1507 represents the sum of the starting and ending terms, the 500 represents the number of terms being added, and the 2 is just a 2.

The generalization of this observation gives the **arithmetic sum formula**.

■ ARITHMETIC SUM FORMULA

If $P_0, P_1, P_2, \ldots$ are the terms of an arithmetic sequence, then

$$P_0 + P_1 + P_2 + \cdots + P_{N-1} = \frac{(P_0 + P_{N-1})N}{2}.$$

The Arithmetic Sum Formula

Informally, the arithmetic sum formula says *to find the sum of consecutive terms of an arithmetic sequence, first add the first and the last terms of the sum, multiply the result by the number of terms being added, and divide by two*.

Population Growth Model

The Logistic Growth Model

The Logistic Growth Model

One of the key tenets of population biology is the idea that there is an inverse relation between the growth rate of a population and its density.

Small populations have plenty of room to spread out and grow, and thus their growth rates tend to be high.

As the population density increases, however, there is less room to grow and there is more competition for resources—the growth rate tends to taper off.

The Logistic Growth Model

Sometimes the population density is so high that resources become scarce or depleted, leading to negative population growth or even to extinction.

The effects of population density on growth rates were studied in the 1950s by behavioral psychologist John B. Calhoun.

The Logistic Growth Model

Calhoun's now classic studies showed that when rats were placed in a closed environment, their behavior and growth rates were normal as long as the rats were not too crowded.

When their environment became too crowded, the rats started to exhibit abnormal behaviors, such as infertility and cannibalism, which effectively put a brake on the rats' growth rate.

In extreme cases, the entire rat population became extinct.

The Logistic Growth Model

Calhoun's experiments with rats are but one classic illustration of the general principle that a *population's growth rate is negatively impacted by the population's density.*

This principle is particularly important in cases in which the population is confined to a limited environment.

Population biologists call such an environment the **habitat**.

The Logistic Growth Model

The habitat might be a cage (as in Calhoun's rat experiments), a lake (for a population of fish), a garden (for a population of snails), and, of course, Earth itself (everyone's habitat).

In 1838, the Belgian mathematician Pierre François Verhulst proposed a mathematical model of population growth for species living within a fixed habitat.

Verhulst called his model the **logistic growth model**.

The Logistic Growth Model

Every biological population living in a confined habitat has a natural intergenerational growth rate that we call the **growth parameter** of that population.

The Logistic Growth Model

The growth parameter of a population depends on the kind of species that makes up the population and the nature of its habitat—a population of beetles in a garden has a different growth parameter than a population of gorillas in the rainforest, and a population of gorillas in the rainforest has a different growth parameter than a population of gorillas in a zoo.

The Logistic Growth Model

Given a specific species and a specific habitat for that species, we will assume the growth parameter is a constant we will denote by *r*.

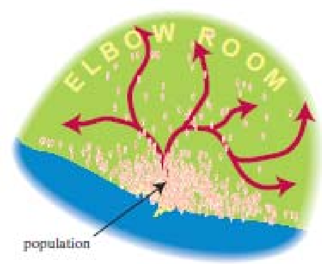
The Logistic Growth Model

The actual growth rate of a specific population living in a specific habitat depends not just on the growth parameter *r* (otherwise we would have an exponential growth model) but also on the amount of "elbow room" available for the population to grow (a variable that changes from generation to generation).

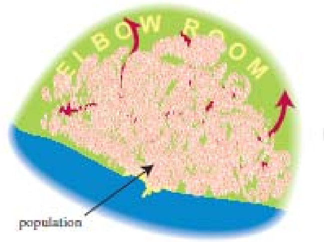
The Logistic Growth Model

When the population is small (relative to the size of the habitat) and there is plenty of elbow room for the population to grow, the growth rate is roughly equal to the growth rate is roughly equal to the growth parameter *r* and the population grows more or less exponentially, as shown by the figure on the next slide.

The Logistic Growth Model

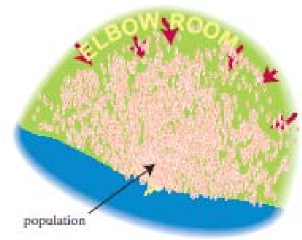


The Logistic Growth Model



As the population gets bigger and there is less space for the population to grow, the growth rate gets proportionally smaller.

The Logistic Growth Model



Sometimes there is a switch to negative growth, and the population starts decreasing for a few generations to get back to a more sustainable level.

Example: A Stable Equilibrium

Fish farming is big business these days, so you decide to give it a try.

You have access to a large, natural pond in which you plan to set up a rainbow trout hatchery.

The carrying capacity of the pond is C = 10,000 fish, and the growth parameter of this type of rainbow trout is $r = 2.5$.

Example: A Stable Equilibrium

We will use the logistic equation to model the growth of the fish population in your pond.

You start by seeding the pond with an initial population of 2000 rainbow trout (i.e., 20% of the pond's carrying capacity, or $p_0 = 0.2$).

Example: A Stable Equilibrium

After the first year (trout have an annual hatching season) the population is given by

$$p_1 = r(1 - p_0)\, p_0 = 2.5 * (1 - 0.2) * (0.2) = 0.4$$

The population of the pond has doubled, and things are looking good!

Example: A Stable Equilibrium

Unfortunately, most of the fish are small fry and not ready to be sent to market.

After the second year the population of the pond is given by

$$p_2 = 2.5 * (1 - 0.4) * (0.4) = 0.6$$

Example: A Stable Equilibrium

The population is no longer doubling, but the hatchery is still doing well.

You are looking forward to even better yields after the third year.

But on the third year you get a big surprise:

$p_3 = 2.5 * (1 - 0.6) * (0.6) = 0.6$

Example: A Stable Equilibrium

Stubbornly, you wait for better luck the next year, but

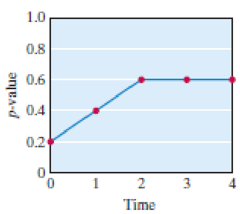
$p_4 = 2.5 * (1 - 0.6) * (0.6) = 0.6$

From the second year on, the hatchery is stuck at 60% of the pond capacity— nothing is going to change unless external forces come into play.

We describe this situation as one in which the population is at a *stable equilibrium*.

Example: A Stable Equilibrium

This figure shows a line graph of the pond's fish population for the first four years.



Population Growth Model

The Exponential Growth Model

The Exponential Growth Model

Before we start a full discussion of exponential growth, we need to spend a little time explaining the mathematical meaning of the term *growth rate*.

In this chapter we will focus on growth rates as they apply to population models, but the concept applies to many other situations besides populations.

The Exponential Growth Model

In the next chapter, for example, we will discuss growth rates again, but in the context of money and finance.

When the size of a population "grows" from some value X to some new value Y, we want to describe the growth in relative terms, so that the growth in going from $X = 2$ to $Y = 4$ is the same as the growth in going from $X = 50$ to $Y = 100$.

Growth Rate

The *growth rate r* of a population as it changes from an initial value *X* (the *baseline*) to a new value *Y* (the *end-value*) is given by the ratio $r = (Y - X)/X$.

(*Note*: It is customary to express growth rates in terms of percentages, so, as a final step, *r* is converted to a percent.)

Growth Rate

One important thing to keep in mind about the definition of growth rate is that it is not symmetric—the growth rate when the baseline is *X* and the endvalue is *Y* is very different from the growth rate when the baseline is *Y* and the end-value is *X*.

Example: The Spread of an Epidemic

In their early stages, infectious diseases such as HIV or the swine flu spread following an exponential growth model— each infected individual infects roughly the same number of healthy individuals over a given period of time.

Formally, this translates into the recursive formula $P_N = (1 + r)P_{N-1}$ where P_N denotes the number of infected individuals in the population at time *N* and *r* denotes the growth rate of the infection.

Example: The Spread of an Epidemic

Every epidemic starts with an original group of infected individuals called "population zero."

Let's consider an epidemic in which "population zero" consists of just one infected individual ($P_0 = 1$) and such that, on average, each infected individual transmits the disease to one healthy individual each month.

Example: The Spread of an Epidemic

This means that every month the number of infected individuals doubles and the growth rate is $r = 1 = 100\%$.

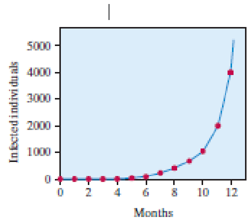
Under this model the number of infected individuals *N* months after the start of the epidemic is given by $P_N = 2^N$.

Example: The Spread of an Epidemic

The figure shows the growth of the epidemic during its first year.

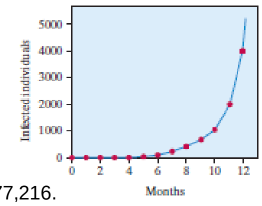
By the end of the first year the number of infected individuals is P_{12} = 212 = 4096.

That's not too bad.



Example: The Spread of an Epidemic

But suppose that no vaccines are found to slow down the epidemic and the growth rate for infected individuals continues at 100% per month. At the end of the second year the number of infected individuals would equal P_{24} = 224 = 16,777,216.



Example: The Spread of an Epidemic

Eight months after that the number of infected individuals would equal P_{32} = 232 4.3 billion (more than half of the world's population); one month later every person on the planet would be infected.

The Exponential Growth Model

The previous example illustrates what happens when exponential growth continues unchecked, and why, when modeling epidemics, exponential growth is a realistic model for a while, but there must be a point in time where the rate of infection has to level off and the model must change.

Otherwise, the human race would have been wiped out many times over.

Geometric Sum Formula

$$P_0 + RP_0 + R^2P_0 + \ldots + R^{N-1}P_0 = \frac{\left(R^{N-1}\right)P_0}{R-1}$$